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ASYMPTOTIC SOLUTION OF A THREE-DIMENSIONAL PROBLEM OF EQUILIBRIUM OF AN ELASTIC BODY WITH A CUT

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A method of reducing the problem of a nonplanar cut to a well investigated problem of a plane cut is given. The method is based on representing the solution in the form of the second potential of the theory of elasticity, and subsequently reducing the problem to a pseudodifferential equation on a nonplanar compact S with the edge ∂S lying in the plane. The resulting equation is then reduced by asymptotic exchange to a pseudodifferential equation on a plane surface which is a projection of S on the plane containing ∂S. The last equation is transformed using a standard method into a sequence of pseudodifferential equations each of which can be solved using the well known methods (e.g. Fourier transforms). If the surface is almost plane, then the first term of the series will already give an approximate solution of the problem.

Solutions of three-dimensional problems of equilibrium of an elastic body with a cut are known for the cases in which the surface S containing the cut lies in a plane [1, 2] or, when the problem is axisymmetric. In the latter case it can be reduced to the problem of conjugation. This was the method used to solve the problem of a cut in the form of a spherical cup [3]. The solution however is cumbersome and does not yield the stress pattern readily.

1. Let us consider an infinite isotropic elastic space with a cut along the surface S which we assume to be a Liapunov manifold of continuous curvature, with the edge ∂S. The cut edges are acted upon by the distributed loads $\mathbf{q}$. Let $\mathbf{n}$ denote the unit vector normal to S at the point x. The points of the cut edges will be accompanied by the plus or minus sign, depending on whether the direction of the normal to the boundary of the elastic body coincides with, or opposes the direction of the normal $\mathbf{n}$ to the surface S. We shall assume that the field $\mathbf{q}$ satisfies the condition

$$\mathbf{q}(x^+) = -\mathbf{q}(x^-) \tag{1.1}$$

at the cut edges. In this case the state of the elastic medium with a cut under a load can be determined by defining a pair $\{S, \mathbf{q}\}$ where $\mathbf{q}(x) = \mathbf{q}(x^+)$ denotes the field on S.

In what follows, we shall denote the displacement and stress fields by $\mathbf{u}$ and $\mathbf{T}_\sigma$ respectively, with $\sigma_n = \mathbf{n}\cdot\mathbf{T}_\sigma$. The dot appearing between the tensors of different order denotes convolution operators in the tensor algebra which are defined on the polyads as follows: $\mathbf{abcde} \to (\mathbf{ab})\cdot(\mathbf{cde}) = (\mathbf{b}\cdot\mathbf{c})\,\mathbf{ade}$; $\mathbf{abcde}$, $\mathbf{ab}$ etc. denote the polyadic (tensor) product of vectors, and $\mathbf{b}\cdot\mathbf{c}$ denotes the scalar product of vectors.

The influence tensor of an unbounded elastic medium is defined by the relation [4]

$$\Gamma(y-z) = \lambda_1 \frac{1}{R}\mathbf{I} + \mu_1 \frac{1}{R^3}\mathbf{RR} \tag{1.2}$$

$$\mathbf{R} = \mathbf{y} - \mathbf{z}, \quad R = |\mathbf{R}|, \quad \lambda_1 = \frac{3 - 4\nu}{16\pi\mu(1-\nu)}, \quad \mu_1 = \frac{1}{16\pi\mu(1-\nu)}$$

where $\mathbf{I}$ is a unit tensor of order two.

Let us express the field $\mathbf{u}$ in terms of the field $\mathbf{b}$ of displacement discontinuities on S

$$\mathbf{b}(x) = \lim_{z \to x-} \mathbf{u}(z) - \lim_{z \to x+} \mathbf{u}(z) \tag{1.3}$$

To do this, we consider [5] the following two states of the elastic medium:

1°. Symmetric forces $\mathbf{q}$ act at the cut edges.

2°. A concentrated force $\mathbf{P}$ acts at the point $z \notin S$ and the forces $\pm\, \sigma_n(y)$ are applied at the cut edges $S^\pm$ so as to prevent the edges from a mutual displacement under the action of the force $\mathbf{P}$. Here $\mathbf{n} = \mathbf{n}(y)$ denotes the normal to the surface at the point $y \in S$.

As we know [4],

$$\sigma_n(y) = \Phi_n(y, z) \cdot \mathbf{P} \tag{1.4}$$

$$\Phi_n(y, z) = \frac{1}{8\pi(1-\nu)R^3}\left[(1-2\nu)(\mathbf{nR} - \mathbf{Rn} - \mathbf{n}\cdot\mathbf{R}\mathbf{I}) - 3\frac{\mathbf{n}\cdot\mathbf{R}}{R^2}\mathbf{RR}\right] \tag{1.5}$$

Applying the theorem of reciprocity of work for these two states and taking into account the fact that the forces of the first state do not perform any work on the displacements corresponding to the second state, we have

$$\mathbf{u}(z)\cdot\mathbf{P} + \int_S [\mathbf{u}(y^+)\cdot\sigma_n(y) + \mathbf{u}(y^-)\cdot(-\sigma_n(y))]\, dS_y = 0 \tag{1.6}$$

This with (1.3), (1.4) and the arbitrariness of the force $\mathbf{P}$, yields

$$\mathbf{u}(\mathbf{b})(z) = \int_S \mathbf{b}(y)\cdot\Phi_n(y, z)\, dS_y \tag{1.7}$$

and from this it follows that the solution of the problem can be sought in the form (1.7).

2. In what follows, we shall need the knowledge of certain properties of the field (1.7) [6,7]. If $\mathbf{b} \in L_p(S)$ and $p > 1$, the angular boundary values of the field (1.7) exist for almost every $x \in S$

$$\lim \mathbf{u}(z) = \mathbf{u}^\pm(x), \quad z \in K(x, \alpha), \quad z \to x^\pm$$

Here ($K(x, \alpha)$ is a cone with the center at the point $x \in S$, the vertex angle $\alpha \in (0, \pi)$ and the axis $\mathbf{n}(x)$). In addition we have the relation

$$\mathbf{u}^\pm(x) = \mp\frac{1}{2}\mathbf{b}(x) + \text{v.p.}\int_S \mathbf{b}(y)\cdot\Phi_n(y, x)\, dS_y \tag{2.1}$$

The principal value of the singular integral

$$\text{v.p.}\int_S = \lim_{\delta \to 0} \int_{S \setminus \dot{S}(x, \delta)}$$

where $S(x, \delta)$ denotes a part of the surface S lying inside the cylinder $C(x, \mathbf{n}(x), \delta)$ of radius δ, with the axis $\mathbf{n}(x)$ passing through the point x. If, in addition, we have $\mathbf{b} \in C^{0,\beta}(S_0)$, $S_0 \subset S$ and $S_0 \cap \partial S = \varnothing$, then the field $\mathbf{u}(\mathbf{b})(z)$ is extended continuously at each interior point S_0 and the boundary values are obtained from (2.1).

Let us introduce the stress tensor operator [6]

$$\mathbf{T}(\mathbf{n}, \nabla) = \mu\mathbf{n}\cdot\nabla\mathbf{I} + \mu(\mathbf{n}\nabla)^* + \lambda\mathbf{n}\nabla \tag{2.2}$$

Its action on the functions with tensor values can be determined using the standard rules of tensor algebra, and ∇ is a vector-differential operator [4], $\mathbf{n}\,\nabla$ is the dyadic product of the unit vector $\mathbf{n}$ and vector operator ∇ ; $(\mathbf{n}\,\nabla)^*$ is the conjugate dyadic operator which acts according to the rule $(\mathbf{n}\,\nabla)^*\,\mathbf{A} = (\mathbf{n}\,(\nabla\,\mathbf{A}))_2$, where the linear operator $(\quad)_2$ on the polyads is obtained as follows: $(\mathbf{abcd})_2 = \mathbf{bacd}$, $(\mathbf{n}\cdot\nabla\,\mathbf{I})\,\mathbf{A} = \mathbf{n}\cdot\nabla\,(\mathbf{I}\,\mathbf{A})$.

Contracting the stress operator with the displacement field we obtain the force vector on the plane with the normal $\mathbf{n}$. Let us compute

$$\mathbf{T}(\mathbf{n}, \nabla)\cdot\mathbf{u}(\mathbf{b})(z) = \int_S \mathbf{T}_z\cdot[\mathbf{b}(y)\cdot\mathbf{\Phi}_\nu(y, z)]\,dS_y, \quad z\notin S$$

Since $\mathbf{h}_n(\mathbf{b}) = \mathbf{T}_z\cdot[\mathbf{b}(y)\cdot\mathbf{\Phi}_\nu(y, z)]$ represents a linear mapping $R^3 \to R^3$, there exists a tensor of order two

$$\mathbf{H}_{n\nu}(y, z) : \mathbf{T}_z\cdot[\mathbf{b}(y)\cdot\mathbf{\Phi}_\nu(y, z)] = \mathbf{b}(y)\cdot\mathbf{H}_{n\nu}(y, z)$$

Here $\nu = \mathbf{n}(y)$ and $\mathbf{n}$ is a unit vector entering the operator $\mathbf{T}_z = \mathbf{T}(\mathbf{n}, \nabla_z)$. It is clear that the tensor $\mathbf{H}_{n\nu}(y, z) = |y - z|^{-3}\,\mathbf{K}_{n\nu}(y, z)$, $\mathbf{K}_{n\nu}(y, z)$ represents a bounded tensor function on $S\times S$.

Thus the force vector on the plane with the normal $\mathbf{n}$ at the point $z\notin S$ in a medium with a cut S on which the displacement field undergoes a discontinuity $\mathbf{b}$, can be written in the form

$$\sigma_n(\mathbf{b})(z) = \int_S \mathbf{b}(y)\cdot\mathbf{H}_{n\nu}(y, z)\,dS_y \tag{2.3}$$

To find $\mathbf{b}$ in terms of the specified force vector $\mathbf{q}$ on S, we must set down the condition (formal for the time being)

$$\lim_{z\to x\pm}\sigma_n(\mathbf{b})(z) = \mathbf{q}(x), \quad \mathbf{n} = \mathbf{n}(x) \tag{2.4}$$

We know [6] that $[\sigma_n(\mathbf{b})]^+ = [\sigma_n(\mathbf{b})]^-$, if $\mathbf{b}\in C^{1,\beta}(S)$, and we also have $\sigma_n(\mathbf{b})\in C^{0,\beta}(R^3\setminus\partial S)$. Obviously, one cannot perform the passage to the limit $z\to x\in S$ directly under the integral sign in (2.3), since the tensor $\mathbf{H}_{n\nu}$ has a singularity R^{-3} and the integral therefore does not exist even in the sense of its principal value. For this reason a suitable regularization must be carried out under the integral appearing in the right hand side of (2.3) when $z\in S$. In the present case the most convenient regularization makes use of transformation of the integral according to the Stokes theorem [7]

$$\sigma_n(\mathbf{b})(z) = \int_S \sigma[\mathbf{n}, \mathbf{H}(y, z), \quad \mathbf{M}(y), \mathbf{b}(y)]\,dS_y + \int_{\partial S}\mathbf{b}(y)\cdot\mathbf{L}(y, z)\,dl_y \tag{2.5}$$

where the following notation is employed:

$$\sigma[\mathbf{n}, \mathbf{H}, \mathbf{M}, \mathbf{b}] = \mu^2[\mathbf{n}\cdot\mathbf{Mb}\cdot\cdot\mathbf{H}_2 + \mathbf{Mb}\cdot\cdot\mathbf{H}_2\cdot\mathbf{n} - \mathbf{n}\cdot\mathbf{H}\cdot\mathbf{M}\cdot\mathbf{b} - \tag{2.6}$$

$$(\mathbf{H}\cdot\mathbf{M}\cdot\mathbf{b})\cdot\mathbf{n}] + \lambda^2\mathbf{n}\mathbf{h}\cdot\mathbf{M}\cdot\mathbf{b} + \lambda\mu\mathbf{n}\mathbf{M}\mathbf{b}\cdots\mathbf{H}_2 + \qquad (2.6)$$
$$\mu(\lambda+\mu)[\mathbf{n}\cdot\mathbf{h}\mathbf{M}\cdot\mathbf{b} + \mathbf{h}(\mathbf{M}\cdot\mathbf{b})\cdot\mathbf{n}]$$
$$\mathbf{H} = \mathbf{H}(y, z) = \nabla_z\boldsymbol{\Gamma}(y-z), \quad \mathbf{h} = \mathbf{h}(y, z) = \nabla_z\cdot\boldsymbol{\Gamma}(y-z)$$
$$\mathbf{M} = \mathbf{M}(y) = \boldsymbol{\nu}\mathbf{D}_y - (\boldsymbol{\nu}\mathbf{D}_y)^* = \boldsymbol{\nu}\nabla_y - (\boldsymbol{\nu}\nabla_y)^*, \quad \boldsymbol{\nu} = \mathbf{n}(y)$$

$\mathbf{D}_y = \nabla_y - \boldsymbol{\nu\nu}\cdot\nabla_y$ is the operaotr of tangential differentiation, $\mathbf{H}_2 = (\mathbf{H}(y, z))_2$ is an isomer of the tensor $\mathbf{H}(y, z)$ defined previously and $\mathbf{L}(y, z)$ is a second order tensor the specific form of which is immaterial.

Using the property of the derivatives of the first potential of the theory of elasticity [6,7] and passing in (2.5) to the limit as $z \to x \in S \setminus \partial S$, we obtain a sum of singular (in the sense of the principal value) of the convolution form integrals

$$\mathbf{n}\int_S \mathbf{H}(\mathbf{M}\mathbf{b})\, dS_y$$

and the contour integral

$$\int_{\partial S} \mathbf{b}(y)\cdot\mathbf{L}(y, x)\, dl_y$$

Now the condition (2.4) has a meaning and defines a pseudodifferential equation on the surface S with edge ∂S. All that follows now aims at performing an accurate change of variables in the singular integrals so as to obtain an equation on a plane, and at solving this equation by means of asymptotic expansions.

3. Let the surface S differ but little from a plane, i.e. let a three-dimensional system of orthogonal unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ exist in which the equation of this surface can be written in the form

$$x_3 = \varepsilon f(x_1, x_2), \quad \varepsilon \in [0, 1) \qquad (3.1)$$

We shall also demand that the edge ∂S of the surface lies in the plane $\Pi : x_3 = 0$, i.e. $f(x_1, x_2) = 0$ when $(x_1, x_2) \in \partial S$.

Let $\pi(A)$ denote the projection of the set A on Π along $\mathbf{e}_3$. The equation of the surface (3.1) generates a mapping $\mathbf{F}$ of the cylinder lying above $\Sigma = \pi(S)$, onto itself. $\mathbf{F} : \Sigma \times R^1 \to \Sigma \times R^1$ acts according to the formula

$$\mathbf{x} = \mathbf{F}(\boldsymbol{\xi}) = \boldsymbol{\xi} + \varepsilon f(\pi\boldsymbol{\xi})\mathbf{e}_3 \qquad (3.2)$$

The mapping $\mathbf{F}$ is in one to one correspondence. Let a field $\varkappa(\mathbf{x})$ be deiined in the neighborhood of the surface S Then $\mathbf{F}$ generates a field $\varkappa_*$ near Σ

$$\varkappa_*(\boldsymbol{\xi}) = \varkappa(\mathbf{x}(\boldsymbol{\xi})) \qquad (3.3)$$

It can be shown that (the symbol $\approx$ denotes equality with an accuracy to terms of order ε^2)

$$(\nabla\varkappa)_*(\boldsymbol{\xi}) \approx \nabla_\xi\varkappa_*(\boldsymbol{\xi}) - \varepsilon\mathbf{p}(\boldsymbol{\xi}_0)\frac{\partial}{\partial\xi_3}\varkappa_*(\boldsymbol{\xi}) \qquad (3.4)$$

$$\xi_0 = \pi(\xi), \quad \mathbf{p}(\xi_0) = \frac{\partial f}{\partial \xi_1}\mathbf{e}_1 + \frac{\partial f}{\partial \xi_2}\mathbf{e}_2 \equiv p_1\mathbf{e}_1 + p_2\mathbf{e}_2 \tag{3.5}$$

In addition we give the following asymptotic expressions to be used below:

$$\mathbf{n}_*(\xi) \approx \mathbf{e}_3 + \varepsilon\mathbf{n}_1(\xi), \quad \mathbf{n}_1(\xi) = -\mathbf{p}(\xi_0) \tag{3.6}$$

$$(\mathbf{D}_x)_* \approx \mathbf{D}_\xi + \varepsilon\left[p_1(\mathbf{e}_3 - \mathbf{e}_1)\frac{\partial}{\partial \xi_1} + p_2(\mathbf{e}_3 - \mathbf{e}_2)\frac{\partial}{\partial \xi_2}\right] \tag{3.7}$$

$$(\mathbf{M}(x))_* \approx \mathbf{M}(\xi) + \varepsilon\mathbf{M}_1(\xi), \quad \mathbf{M}_*\mathbf{b}_* = (\mathbf{M}\mathbf{b})_* \tag{3.8}$$

$$\mathbf{M}_1(\xi) = (\mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2)\left[p_1\frac{\partial}{\partial \xi_2} - p_2\frac{\partial}{\partial \xi_1}\right]$$

Let $\xi \to \mathbf{x}$, $\eta \to \mathbf{y}$, $\mathbf{R} = \mathbf{y} - \mathbf{x}$, $\mathbf{r} = \eta - \xi$, $\rho = \pi(\mathbf{R})$ be the projection of $\mathbf{R}$ on Π, $\Delta(\eta, \xi) = f(\eta_0) - f(\xi_0)$, $\eta_0 = \pi(\eta)$.
From (3.2) it follows that

$$\mathbf{R} = \mathbf{r} + \varepsilon\Delta\mathbf{e}_3 \tag{3.9}$$

$$R^\alpha \approx r^\alpha + \alpha\varepsilon\Delta r^{\alpha-1}\mathbf{r}_1\cdot\mathbf{e}_3, \quad \mathbf{r}_1 = r^{-1}\mathbf{r}, \quad \alpha \in \mathbf{R}^1$$

For the tensor $\mathbf{H}(y, x) = \nabla_x \mathbf{\Gamma}(y - x)$ we have (the summation from 1 to 3 is carried out over the umbral index)

$$\mathbf{H}(y, x) = \left\{\lambda_1\mathbf{R}\mathbf{I} - \mu_1(\mathbf{I}\mathbf{R} + \mathbf{e}_j\mathbf{R}\mathbf{e}_j) + 3\mu_1\frac{1}{R^2}\mathbf{R}\mathbf{R}\mathbf{R}\right\}\frac{1}{R^3}$$

Applying Eqs. (3.9), we obtain

$$\mathbf{H}_*(\eta, \xi) \approx \mathbf{H}(\eta, \xi) + \varepsilon\mathbf{H}_1(\eta, \xi) + \varepsilon\mathbf{r}_1\cdot\mathbf{e}_3\mathbf{T}(\eta, \xi)$$

$$\mathbf{H}_1(\eta, \xi) = \Delta r^{-3}\{\lambda_1\mathbf{e}_3\mathbf{I} - \mu_1(\mathbf{I}\mathbf{e}_3 + \mathbf{e}_j\mathbf{e}_3\mathbf{e}_j) + 3\mu_1 r^{-2}(\mathbf{r}\mathbf{r}\mathbf{e}_3)\}$$

where $\mathbf{T}(\eta, \xi)$ is a pointwise bounded tensor function and $(\mathbf{r}\mathbf{r}\mathbf{e}_3) = \mathbf{r}\mathbf{r}\mathbf{e}_3 + \mathbf{r}\mathbf{e}_3\mathbf{r} + \mathbf{e}_3\mathbf{r}\mathbf{r}$.
When $x, y \in S$, $\xi, \eta \in \Sigma$, we have $\mathbf{r} = \rho$ and $\mathbf{r}_1\cdot\mathbf{e}_3 = 0$, therefore

$$\mathbf{H}_*(\eta, \xi) \approx \mathbf{H}(\eta, \xi) + \varepsilon\mathbf{H}_1(\eta, \xi) \tag{3.10}$$

The integrand expressions in (2.5) include the fields $\mathbf{M}(\mathbf{n}(y), \nabla_y)\,\mathbf{b}(y)$ for $y \in S$. To compute these fields we must, in addition, define the field $\mathbf{b}$ near the surface S. We shall do this as follows (π_S is the projection onto S along $\mathbf{e}_3$):

$$\mathbf{b}(z) = \mathbf{b}(\pi_S z), \quad \forall z \in \Sigma \times R^1$$

According to (3.3), a field $\mathbf{b}_*(\xi)$ appears in the neighborhood of Σ and it is clear that $\mathbf{b}_* = \mathrm{const}$ along the normal $\mathbf{e}_3$ to the surface Σ. Since $\mathbf{M}(\mathbf{n}, \nabla) = \mathbf{M}(\mathbf{n}, \mathbf{D})$, it follows from the Hadamard – Hugoniot theorem [8], which states that $\mathbf{D}f_1|_S = \mathbf{D}f_2|_S$ for any f_1 and $f_2 : f_1|_S = f_2|_S$, that any supplementary smooth definition of the fields near the surface is correct.

Let us now consider the singular integral in the sense of its principal value

$$\text{v.p.}\int_S K(x, y)\,\varphi(y)\,dS_y = \lim_{\delta\to 0}\int_{S\setminus S(x,\delta)} K(x, y)\,\varphi(y)\,dS_y \tag{3.11}$$

Performing the change of variables (3.2) in the integral following the limit sign in (3.11) which represents an ordinary nonsingular integral we obtain

$$\int\limits_{S\setminus S(x,\,\delta)} K(x,y)\,\varphi(y)\,dS_y = \int\limits_{\Sigma\setminus\pi S(x,\,\delta)} K_*(\xi,\eta)\,\varphi_*(\eta)\,I(\eta)\,d\eta \qquad (3.12)$$

Here $I(\eta)$ denotes the first quadratic form of the surface S. Passing in (3.12) to the limit as $\delta \to 0$, we obtain

$$\text{v.p.}\int\limits_S K(x,y)\,\varphi(y)\,dS_y = \fint\limits_\Sigma K_*(\xi,\eta)\,\varphi_*(\eta)\,I(\eta)\,d\eta$$

and the singular integral in the right hand side is not equal to the corresponding integral in the sense of its principal value, since the region $\pi S(x,\delta)$ is not a circle. Nevertheless we have [6, 9]

$$\fint\limits_\Sigma K_*(\xi,\eta)\,\varphi_*(\eta)\,I(\eta)\,d\eta = \text{v.p.}\int\limits_\Sigma K_*(\xi,\eta)\,\varphi_*(\eta)\,I(\eta)\,d\eta -$$

$$\varphi_*(\xi)\,I(\xi)\int\limits_0^{2\pi} k_*(\xi,\vartheta)\ln\cos\alpha(\xi,\vartheta)\,d\vartheta$$

where

$$k_*(\xi,\vartheta) = \lim_{|\eta-\xi|\to 0} |\eta-\xi|^2 K_*(\xi,\eta)$$

and $\eta \to \xi$ in such a manner that the angle between the vectors $\eta - \xi$ and ξ is constant and $\alpha(\xi,\vartheta)$ denotes the angle between the vector $\mathbf{R} = \mathbf{y} - \mathbf{x}$ and its projection $\boldsymbol{\rho}$ on Σ.

It can be shown that [6]

$$\cos\alpha(\xi,\vartheta) = n_3(\xi)\,[n_3^2 + (n_1\cos\vartheta + n_2\sin\vartheta)^2]^{-1/2}$$

Expanding $\cos\alpha(\xi,\vartheta)$ into a series in powers of ε, we find that $\cos\alpha(\xi,\vartheta) \approx 1$ with the accuracy of up to ε^2. Therefore we have

$$\fint\limits_\Sigma \approx \text{v.p.}\int\limits_\Sigma$$

and since $I(\eta) \approx 1$, from (3.12) it follows that

$$\Big(\text{v.p.}\int\limits_S K(x,y)\,\varphi(y)\,dS_y\Big)_* \approx \text{v.p.}\int\limits_\Sigma K_*(\xi,\eta)\,\varphi_*(\eta)\,d\eta$$

We can now perform the change of variables in (2.5). From (3.10) we obtain

$$(\mathbf{H}_*(\eta,\xi))_2 \approx (\mathbf{H}(\eta,\xi))_2 + \varepsilon\,(\mathbf{H}_1(\eta,\xi))_2$$
$$(\nabla_x\cdot\mathbf{\Gamma}(x-y))_* \approx \mathbf{h}(\eta,\xi) + \varepsilon\mathbf{h}_1(\eta,\xi)$$
$$\mathbf{h}(\eta,\xi) = \nabla_\xi\cdot\mathbf{\Gamma}(\eta-\xi) = (\lambda_1-\mu_1)\,r^{-3}\mathbf{r}$$
$$\mathbf{h}_1(\eta,\xi) = (\lambda_1-\mu_1)\,r^{-3}\Delta\mathbf{e}_3$$

Substituting these expressions together with (3.6) and (3.8) into (2.5), we obtain

$$
\begin{aligned}
&(\boldsymbol{\sigma}_n(\mathbf{b}))_*(\xi) \approx \boldsymbol{\sigma}_0(\mathbf{b}_*)(\xi) + \varepsilon\boldsymbol{\sigma}_1(\mathbf{b}_*)(\xi) + \mathbf{L}_*(\mathbf{b}_*)(\xi) \\
&\boldsymbol{\sigma}_0(\mathbf{b}_*)(\xi) = \int_{\Sigma} \boldsymbol{\sigma}[\mathbf{e}_3, \mathbf{H}(\eta, \xi), \mathbf{M}(\eta), \mathbf{b}_*(\eta)]\, d\eta \\
&\boldsymbol{\sigma}_1(\mathbf{b}_*)(\xi) = \int_{\Sigma} \{\boldsymbol{\sigma}[\mathbf{n}_1, \mathbf{H}(\eta, \xi), \mathbf{M}(\eta), \mathbf{b}_*(\eta)] + \\
&\quad \boldsymbol{\sigma}[\mathbf{e}_3, \mathbf{H}_1(\eta, \xi), \mathbf{M}(\eta), \mathbf{b}_*(\eta)] + \boldsymbol{\sigma}[\mathbf{e}_3, \mathbf{H}(\eta, \xi), \mathbf{M}_1(\eta), \mathbf{b}_*(\eta)]\}\, d\eta \\
&\mathbf{L}_*(\mathbf{b}_*)(\xi) = \int_{\partial\Sigma} \mathbf{b}_*(\eta)\cdot\mathbf{L}_*(\eta, \xi)\, dl_\eta
\end{aligned}
\qquad (3.13)
$$

The vector $\boldsymbol{\sigma}[\mathbf{n}, \mathbf{H}, \mathbf{M}, \mathbf{b}]$ is defined by (2.6), and all operators appearing in (3.13) are obviously linear.

4. We shall seek a solution of the pseudo-differential (integro-differential) equation

$$(\boldsymbol{\sigma}_n(\mathbf{b}))_* \equiv (\boldsymbol{\sigma}_n)_*(\mathbf{b}_*) = \mathbf{q}_* \qquad (4.1)$$

as a formal series in ε

$$\mathbf{b}_*(\xi) = \mathbf{b}_0(\xi) + \varepsilon\mathbf{b}_1(\xi) + \ldots \qquad (4.2)$$

We note that the equation (4.1) is equivalent to (2.5), since the mapping is in one to one correspondence and smooth.

Expanding the lead $\mathbf{q}_*(\xi) = \mathbf{q}_0(\xi) + \varepsilon\,\mathbf{q}_1(\xi) + \ldots$ into a series in powers of ε, substituting (4.2) into (3.13) and equating terms of like power in ε, we obtain

$$\boldsymbol{\sigma}_0(\mathbf{b}_0) + \mathbf{L}_*(\mathbf{b}_0) = \mathbf{q}_0 \quad \text{problem } J_0)$$

$$\boldsymbol{\sigma}_0(\mathbf{b}_1) + \mathbf{L}_*(\mathbf{b}_1) = \mathbf{q}_1 \quad \text{problem } J_1)$$

Let us consider the problem J_0. From the definition of $\boldsymbol{\sigma}_0$ it follows that the problem is that of the cut in Σ with the load $\mathbf{q}_0$. If the solution $\mathbf{b}_0$ of the problem is sought in the class of fields in which the energy density is integrable in the neighborhood of $\partial\Sigma$, we arrive at the physically obvious condition $\mathbf{b}_0|_{\partial\Sigma} = 0$. and the integral over $\partial\Sigma$ vanishes, i.e. $\mathbf{L}_*(\mathbf{b}_0) = 0$.
Having determined the solution $\mathbf{b}_0$ of the problem of a plane cut we substitute it into the problem J_1 and we obtain an equation for determining $\mathbf{b}_1$. Thus J_1 represents a problem of a plane cut in Σ subjected to the forces $\mathbf{Q}_1 = \mathbf{q}_1 - \boldsymbol{\sigma}_1(\mathbf{b}_0)$.
Performing the expansion in ε to the accuracy of ε^{n+1} and equating the expressions accompanying the like powers of ε, we would obtain a sequence of problems $J_0, J_1, \ldots, J_n$. A solution $\mathbf{b}_k$ of a problem J_k can be obtained provided that the solutions $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{k-1}$ of the problems $J_0, J_1, \ldots, J_{k-1}$ are known. The operators of the problems J_k are identical for all k, only the right-hand sides $\mathbf{Q}_k$, i.e. the fictitious, differ.
It can be proved that

$$\mathbf{b}(x) - \sum_{k=0}^{n} \varepsilon^k \mathbf{b}_k(\xi(x)) = O(\varepsilon^{n+1})$$

therefore if the surface is nearly plane, then $\mathbf{b}_0 + \varepsilon\mathbf{b}_1$ represents an approximate solution of the problem.

We note that the above asymptotic method of solving a pseudodifferential equation on nonplanar manifolds with an edge can be applied, after obvious modifications, to solving the problems of cuts under the conditions of plane deformation. Since the solutions are simple for the case of a rectilinear cut, the method is also effective in the case of two-dimensional problems.

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